JOURNAL OF DIFFERENTIAL EQUATIONS **154**, 454–469 (1999)
Article ID jdeq.1998.3574, available online at http://www.idealibrary.com on IDEAL®

A Class of Second-Order Differential Equations

Michael Plum

Universität Karlsruhe, D-76128 Karlsruhe, Germany

and

Raymond M. Redheffer

University of California, Los Angeles, California 90024

Received July 22, 1998

Throughout this paper I denotes an interval of the real line, specialized to be open or to be $[0, \infty)$ as the case may be. The most general differential equations with which we will be concerned are

$$(pr')' + qr = \frac{c^2}{pr^3}, \qquad (py')' + qy = 0 \tag{ac}$$

where $p, q \in C(I)$, $p \neq 0$, and c is a nonzero constant.

Theorem 1 develops a connection between these two equations under the assumption that all functions are complex. In Theorem 2 we specialize to the real case and study asymptotic behavior on $[0, \infty)$. Further specialization to the case $p = 1$ follows. The latter has points of contact with theorems of Hartman and others, as mentioned in due course. Some concluding remarks pertain again to the general case (ac).

The letters α, β, γ, δ, μ, ν, τ, θ introduced without explanation denote constants. To simplify notation we often use f as an abbreviation for $f(t)$. This use is clear from the context. For example, $\liminf_{t \to \infty} \sqrt{q}\,(u^2 + v^2) \leqslant c$ is an abbreviation for

$$\liminf_{t \to \infty} \sqrt{q(t)}\,(u(t)^2 + v(t)^2) \leqslant c.$$

The Wronskian is denoted by $W(u, v) = uv' - u'v$. The classes of continuous and differentiable functions on I are denoted by $C(I)$ and $D(I)$, respectively. For $j \geqslant 1$, $y \in C^j(I) \Leftrightarrow y^{(j)} \in C(I)$. All differential equations hold

0022-0396/99 $30.00

pointwise, that is, for $t \in I$. General solutions are allowed but require no new ideas and will not be emphasized here.

1. EQUATIONS WITH COMPLEX COEFFICIENTS

In this section only, all functions p, q, u, v, r, s are complex. The following theorem is proved for real functions in [8], under the additional hypothesis that p is differentiable. The proof given here is entirely different from that in [8]. For related results and historical background see [3, 5, 6].

THEOREM 1. *Let $p, q \in C(I)$, $p \neq 0$, and let c be a nonzero constant. We denote by* (abcd) *the respective equations*

$$(pr')' + qr = \frac{c^2}{pr^3}, \qquad pr^2 s' = c, \quad (py')' + qy = 0, \quad pW(u, v) = c. \qquad \text{(abcd)}$$

(i) *If* (ab) *hold,* $u = r \cos s$, *and* $v = r \sin s$ *satsify* $u^2 + v^2 \neq 0$ *and* (cd).

(ii) *If* u, v *satisfy* $u^2 + v^2 \neq 0$ *and* (cd), *they can be written in the form* $u = r \cos s$, $v = r \sin s$ *where* r, s *satisfy* (ab).

If u and v are linearly independent solutions of (c) then (d) holds for some $c \neq 0$ automatically, by Abel's theorem. This leads to a corresponding reformulation of (ii).

LEMMA 1. *Let $u, v \in D(I)$ be complex-valued functions with $u^2 + v^2 \neq 0$. Then there are complex-valued functions $r, s \in D(I)$ such that $u = r \cos s$, $v = r \sin s$, and $r \neq 0$.*

Proof. We can write $u(t)^2 + v(t)^2 = \xi(t) + i\eta(t) = \rho(t)\, e^{i\sigma(t)}$ for $t \in I$, where ξ, η, ρ, σ are real and $\rho > 0$. Also $\rho, \sigma \in D(I)$. We define $r = \sqrt{\rho(t)}\, e^{i\sigma(t)/2}$. Then $r \in D(I)$ and $r^2 = u^2 + v^2$. The functions $U = u/r$, $V = v/r$ satisfy U, $V \in D(I)$ and

$$(U + iV)(U - iV) = U^2 + V^2 = 1. \tag{1}$$

Since $U + iV \neq 0$ this function has a continuous logarithm which we write as $is = is(t)$. Thus

$$U + iV = e^{is}, \qquad U - iV = e^{-is},$$

where the second equation follows from (1). Addition and subtraction give $U = \cos s$, $V = \sin s$. Lemma 1 follows from $u = rU$, $v = rV$.

LEMMA 2. *If* $u = r\cos s$ *and* $v = r\sin s$, *where* pr' *and* ps' *are in* $D(I)$, *then* pu' *and* pv' *are in* $D(I)$ *and the equations* $(pu')' + qu = 0$, $(pv')' + qv = 0$ *are equivalent respectively to*

$$((pr')' + qr - pr(s')^2)\cos s = \frac{(r^2ps')'}{r}\sin s \tag{2}$$

and to (2) *with* $\sin s$ *on the left,* $-\cos s$ *on the right.*

Proof. We have $pu' = pr'\cos s - (ps')r\sin s$, hence $pu' \in D(I)$, and similarly for pv'. The result follows by substitution, taking care to keep the grouping pu', pv', pr', ps'.

Proof of Theorem 1(i). We have $u^2 + v^2 = r^2$ which is nonzero by (a). Equation (a) implies $pr' \in C^1(I)$ and (b) implies $ps' \in D(I)$. Hence we can use Lemma 2. Condition (b) shows that the coefficient of $\sin s$ in (2) is 0, and (ab) show that the coefficient of $\cos s$ is also 0. Hence u and v satisfy (c). The Wronskian $W(u, v)$ is

$$\begin{vmatrix} r\cos s & r\sin s \\ -rs'\sin s + r'\cos s & rs'\cos s + r'\sin s \end{vmatrix} = \begin{vmatrix} r\cos s & r\sin s \\ -rs'\sin s & rs'\cos s \end{vmatrix} = r^2 s'.$$

Equation (b) yields $pW(u, v) = c$, completing the proof.

Proof of Theorem 1(ii). Lemma 1 gives $u = r\cos s$, $v = r\sin s$ where $r, s \in D(I)$ and $r \neq 0$. Since $pu', pv' \in C^1(I)$, the equation $r^2 = u^2 + v^2$ gives $pr' \in C^1(I)$. Using this and pu', pv' again we find that $ps'\sin s$ and $ps'\cos s$ are both in $C^1(I)$. Since $\cos s$ and $\sin s$ do not vanish simultaneously, $ps' \in C^1(I)$. Therefore (2) holds by Lemma 2. In view of (d), the above calculation of $W(u, v)$ yields $pr^2 s' = c$. Hence the right side of (2) is 0, and the equations associated with u and v become

$$\left((pr')' + qr - \frac{c^2}{pr^3}\right)\cos s = 0, \qquad \left((pr')' + qr - \frac{c^2}{pr^3}\right)\sin s = 0,$$

respectively. Since $\cos s$ and $\sin s$ do not vanish simultaneously, r satisfies (a). This completes the proof.

2. REAL FUNCTIONS

From now on all functions p, q, r, s, u, v are real and $I = [0, \infty)$. We consider the equations (ac) of Theorem 1, namely

$$(pr')' + qr = \frac{c^2}{pr^3}, \qquad (py')' + qy = 0. \tag{ac}$$

THEOREM 2. *Let r satisfy* (a) *where $p, q \in C(I)$ and $p, q, c > 0$. Suppose further that*

$$\int_0^\infty \frac{ds}{p(s)} = \infty, \qquad \limsup_{t \to \infty} \sqrt[4]{pq} \int_0^t \frac{ds}{p(s)} = \infty. \tag{3}$$

Then $\limsup_{t \to \infty} \sqrt[4]{pq}\, r \geqslant \sqrt{c}$.

COROLLARY 1. *If* (3) *holds, Eq.* (c) *cannot have two linearly independent solutions u, v both of which are $o((pq)^{-1/4})$ as $t \to \infty$.*

Proof of Theorem 2. If the conclusion fails then $\sqrt[4]{pq}\, r \leqslant \theta\sqrt{c}$ for $t \gg 1$, where $\theta < 1$. Hence

$$(pr')' = \frac{c^2 - pqr^4}{pr^3} \geqslant \frac{\beta}{pr^3}, \qquad t \gg 1, \tag{4}$$

where $\beta = c^2(1 - \theta^4) > 0$. It follows that pr' is increasing for large t and hence $\lim_{t \to \infty} pr' = L \leqslant \infty$ exists.

Case 1. If $L > 0$ we have $pr' \geqslant \gamma > 0$ for $t \gg 1$, hence

$$r \geqslant \gamma \int_0^t \frac{ds}{p(s)} - \mu.$$

Divergence of the integral ensures

$$\sqrt[4]{pq}\, r \geqslant \sqrt[4]{pq} \left(\int_0^t \frac{\gamma}{p(s)}\, ds - \mu \right) \geqslant \frac{\sqrt[4]{pq}}{2} \int_0^t \frac{\gamma}{p(s)}\, ds$$

for $t \gg 1$, so $\limsup \sqrt[4]{pq}\, r = \infty$. This contradicts the initial assumption.

Case 2. If $L \leqslant 0$ then $pr' \leqslant 0$ for large t. (For $L = 0$ this follows from the fact that pr' is increasing by (4)). Hence $r' \leqslant 0$, so $r \leqslant v$ for some constant v. Inequality (4) gives $(pr')' \geqslant \delta/p$ where $\delta = \beta/v^3$. Therefore

$$pr' \geqslant \delta \int_0^t \frac{ds}{p(s)} - O(1),$$

contradicting the fact that $pr' \leqslant L$ for large t.

Proof of Corollary 1. If there are two such solutions u, v, Theorem 1 can be used to construct $r = o(pq)^{-1/4}$. Since $|c| > 0$ for independent u, v, this contradicts Theorem 2.

If $p=1$ and $\limsup_{t\to\infty} t^4 q=\infty$, the corollary says that $y''+qy=0$ cannot have two linearly independent solutions both of which are $o(q^{-1/4})$ as $t\to\infty$. The following remark pertains to this case.

Remark 1. Let $q>0$, $q\in C(I)$ and $r''+qr=c^2/r^3$ with $c\neq 0$. Then $\liminf_{t\to\infty} t^2 q=\infty \Rightarrow \liminf_{t\to\infty} r^4 q\leqslant c^2$.

For proof, suppose the conclusion fails. Then $r^4 q\geqslant \theta c^2$ for $t\gg 1$, where $\theta>1$. The differential equation gives

$$r''+fr=0, \qquad f=q\left(1-\frac{c^2}{qr^4}\right)\geqslant q\left(1-\frac{1}{\theta}\right), \quad t\gg 1. \tag{5}$$

Since $\liminf_{t\to\infty} t^2 f=\infty$, Eq. (5) is oscillatory, which contradicts the fact that $r\neq 0$.

When applied to $y''+qy=0$, Remark 1 implies that if u, v are two solutions with $W(u,v)=c>0$, then

$$\liminf_{t\to\infty} t^2 q=\infty \Rightarrow \liminf_{t\to\infty} \sqrt{q}\,(u^2+v^2)\leqslant c.$$

3. ESTIMATION OF A PRODUCT

Continuing with the case $p=1$, we will prove:

THEOREM 3. *On $I=[0,\infty)$ suppose $q>0$, $q\in C^1(I)$ and suppose that the equation $y''+qy=0$ is oscillatory. Let u, v be two solutions of this equation with $W(u,v)=c$. Then*

$$\left(\limsup_{t\to\infty} q^{1/4}u\right)\left(\limsup_{t\to\infty} q^{1/4}v\right)\geqslant |c|\left(\liminf_{t\to\infty}\frac{A}{B}\right)^{1/2},$$

where $A(t)=\min\{q(s): t\leqslant s\leqslant t+\pi/\sqrt{q(t)}\}$ and $B(t)=\max q(s)$ on the same interval.

If we define

$$a=\limsup_{t\to\infty} q^{1/4}u, \qquad b=\limsup_{t\to\infty} q^{1/4}v, \quad d^2=\liminf_{t\to\infty}\frac{A}{B}$$

the conclusion is $ab\geqslant |cd|$. This notation is used in the proof and also in the following corollary:

Corollary 2. *Under the hypothesis of Theorem 3 suppose* $d > 0$. *Then if* $u = O(q^{-1/4})$, *it is impossible to have a linearly independent solution* $v = o(q^{-1/4})$.

In all cases $0 \leqslant d \leqslant 1$ and for a large class of functions q we have $d = 1$. Comparing Corollaries 1 and 2, we note that the former asserts the impossibilty of $u = o(q^{-1/4})$, $v = o(q^{-1/4})$ for independent solutions u, v while the latter asserts the impossibility of $u = O(q^{-1/4})$, $v = o(q^{-1/4})$.

In the following discussion we assume u, v, q as in Theorem 3 and, without loss of generality, $W(u, v) = c < 0$. Thus $-c = |c|$. Since the conclusion is trivial when $d = 0$ we assume $d > 0$.

Lemma 3. *Let* $U = q^{1/4}u$ *and* $V = q^{1/4}v$. *Then*

$$U(\rho) = 0 \Rightarrow U'(\rho)\, V(\rho) = |c| \sqrt{q(\rho)}.$$

This follows from $W(wu, wv) = w^2 W(u, v)$, $w \in C^1$.

Lemma 4. *Let* $u'' + qu = 0$, $u(\rho) = 0$, $u'(\rho) > 0$, *and let* σ *be the first zero of* u *beyond* ρ. *Suppose that* $\tilde{q}$ *is some constant satisfying* $\tilde{q} \geqslant q$ *for* $\rho \leqslant t \leqslant \rho + \pi/\sqrt{q(\rho)}$ *and* $\tilde{u}'' + \tilde{q}\tilde{u} = 0$, $\tilde{u}(\rho) = 0$, $\tilde{u}'(\rho) = u'(\rho)$. *Then the first zero* $\tilde{\sigma}$ *of* $\tilde{u}$ *beyond* ρ *satisfies* $\tilde{\sigma} \leqslant \sigma$ *and* $\tilde{u}(t) \leqslant u(t)$ *on* $(\rho, \tilde{\sigma})$.

Proof. This is essentially equivalent to Sturm's comparison theorem in one of its many forms. Here we give an independent proof using differential inequalities. Let (ρ, τ) denote the subinterval of (ρ, σ) on which both u and $\tilde{u}$ are positive. Thus $\tau = \sigma$ if $\tilde{u}(t) > 0$ on (ρ, σ) and otherwise $\tau = \tilde{\sigma}$. In particular,

$$\tau \leqslant \tilde{\sigma} = \rho + \frac{\pi}{\sqrt{\tilde{q}}} \leqslant \rho + \frac{\pi}{\sqrt{q(\rho)}}$$

so that $\tilde{q} \geqslant q$ on $[\rho, \tau]$. We define w as on the left below and compute w':

$$w = \begin{vmatrix} u & \tilde{u} \\ u' & \tilde{u}' \end{vmatrix}, \qquad w' = \begin{vmatrix} u & \tilde{u} \\ -qu & -\tilde{q}\tilde{u} \end{vmatrix} = (q - \tilde{q})\, u\tilde{u}.$$

Hence $w' \leqslant 0$ on (ρ, τ). This gives $w \leqslant w(\rho) = 0$, or in other words $u\tilde{u}' \leqslant u'\tilde{u}$ on (ρ, τ). Dividing by $u\tilde{u}$ we find that $\log(\tilde{u}/u)$, hence $\tilde{u}/u$, is decreasing on (ρ, τ). By l'Hospital's rule

$$\lim_{t \to \rho} \frac{\tilde{u}(t)}{u(t)} = \lim_{t \to \rho} \frac{\tilde{u}'(t)}{u'(t)} = 1.$$

Therefore $\tilde{u}/u \leqslant 1$ on (ρ, τ), which gives both conclusions in Lemma 4.

Proof of Theorem 3. We work on the interval (ρ, σ) of Lemma 4, with u as in that lemma. The functions U, V are as in Lemma 3 and we choose $\alpha > a$, $\beta > b$, $0 < \delta < d^2$ in the notation following the theorem. Since the equation is oscillatory we can, and do, take ρ so large that for $t \geqslant \rho$

$$q^{1/4}(t)\, u(t) < \alpha, \qquad q^{1/4}(t)\, v(t) < \beta, \quad \frac{A(\rho)}{B(\rho)} > \delta.$$

With $\omega^2 = B(\rho)$ define

$$\tilde{u}(t) = \frac{u'(\rho)}{\omega} \sin \omega(t - \rho), \qquad \rho \leqslant t \leqslant \sigma.$$

Then $\tilde{u}'' + B(\rho)\, \tilde{u} = 0$, $\tilde{u}(\rho) = 0$, $\tilde{u}'(\rho) = u'(\rho)$. The definition of A and B yields

$$A(\rho) \leqslant q(t) \leqslant B(\rho) \qquad \text{for} \quad \rho \leqslant t \leqslant \rho + \frac{\pi}{\sqrt{q(\rho)}}$$

and hence Lemma 4 yields, for $\tilde{\sigma} := \rho + \pi/\omega$,

$$\max_{\rho \leqslant t \leqslant \tilde{\sigma}} u(t) \geqslant \max_{\rho \leqslant t \leqslant \tilde{\sigma}} \tilde{u}(t) = \frac{u'(\rho)}{\omega} = \frac{u'(\rho)}{B(\rho)^{1/2}},$$

while $\alpha > a$ gives, on the same interval,

$$\max u(t) = \max q^{-1/4} q^{1/4} u \leqslant A(\rho)^{-1/4} \alpha.$$

These two relations together yield

$$\alpha \geqslant u'(\rho)\, \frac{A(\rho)^{1/4}}{B(\rho)^{1/2}}.$$

On the other hand $\beta > b$ yields, by Lemma 3,

$$\beta > V(\rho) = \frac{|c| \sqrt{q(\rho)}}{U'(\rho)} = \frac{|c| \sqrt{q(\rho)}}{q^{1/4}(\rho)\, u'(\rho)} = \frac{|c|\, q^{1/4}(\rho)}{u'(\rho)} \geqslant \frac{|c|\, A(\rho)^{1/4}}{u'(\rho)}.$$

Here we used $A(\rho) \leqslant q(\rho)$. By the last two inequalities,

$$\alpha\beta > |c|\, \frac{A(\rho)^{1/2}}{B(\rho)^{1/2}} > |c| \sqrt{\delta}.$$

Letting $\alpha \to a$, $\beta \to b$, $\delta \to d^2$ we get Theorem 3.

4. A COUNTEREXAMPLE OUTLINED

If $q \in C$ and $q(t) \to \infty$ as $t \to \infty$, the equation $y'' + qy = 0$ cannot have two linearly independent solutions u, v both of which are $o(q^{-1/4})$ as $t \to \infty$. This follows from Corollary 1. The question whether one nontrivial solution $u = o(q^{-1/4})$ can exist was for a time open, and was then answered affirmatively (in the second author's 1997 Karlsruhe lectures) by use of Theorem 1. Without computational details, an outline of the procedure is given here.

The general idea is as follows. By choice of q we construct a function $r > 0$ and a sequence of points $\{t_i\}$ such that

$$r'' + qr = \frac{1}{r^3}, \qquad \int_{t_i}^{t_{i+1}} \frac{dt}{r(t)^2} = \pi.$$

On a subinterval $J_i = (t_i + \varepsilon_i, t_{i+1} - \eta_i)$ of (t_i, t_{i+1}) we impose the condition $qr^4 = \delta_i$, where δ_i are positive constants with $\delta_i \to 0$. In the small intervals around t_i we interpolate by means of two polynomials, one on each side of t_i, to get a smooth solution r. The desired solution $u = o(q^{-1/4})$ is given by

$$u(t) = r(t) \sin s(t), \qquad s(t) = \int_{t_0}^{t} \frac{dt}{r^2(t)}.$$

That $u'' + qu = 0$ follows from Theorem 1. Since $s(t_i) = \pi i$ the failure of $r = o(q^{-1/4})$ near t_i does not prevent the validity of $u = o(q^{-1/4})$.

By [5], the function r on J_i has the form

$$r = (\alpha + 2\beta t + \gamma t^2)^{1/2}, \qquad \alpha, \beta, \gamma \text{ constant}.$$

The condition $qr^4 = \delta$ holds if $\alpha\gamma - \beta^2 = 1 - \delta$ and gives q. With the aid of these specific formulas for r and q on J_i, the construction of an example can be completed along the lines indicated above and one can even have $q \in C^n$ for any given n.

5. GENERALIZATION OF SONIN'S THEOREM

In [4] it was seen that the solutions of $u'' + qu = 0$ often show a simple behavior of $y = q^{1/4}u$ at its relative extrema $y(t_k)$. The proof depends on an extension of Sonin's theorem. Here we apply a similar technique to r, using the notation $z\downarrow$ to mean that z is weakly decreasing. Our formulations assume a sequence of isolated extrema; if the functions have intervals of constancy, it is left to the reader to provide the necessary changes.

LEMMA 5. *On an open interval I let $P, R, S \in C^1$ and $Q \in C^0$. Suppose further that $P' \leqslant 2Q$, $R' \leqslant 0$, $S' \leqslant 0$ and that*

$$Py'' + Qy' + Ry = \frac{S}{y^3}, \qquad z = P(y')^2 + Ry^2 + \frac{S}{y^2}.$$

Then $z\downarrow$, hence $(Ry^2 + S/y^2)\downarrow$ on the sequence $\{t_k\}$ where y has interior maxima and minima.

For the *proof*, compute z' and substitute for Py'' the expression given by the differential equation. The result after simplification is

$$z' = (y')^2 (P' - 2Q) + R'y^2 + \frac{S'}{y^2}.$$

This gives the first statement and the second follows because $y' = 0$ at the interior extrema of y. The special case $y'' + Qy = 0$ is equivalent to a well-known theorem of Sonin [7, p. 164]. We make two remarks.

Remark 2. If the differential equation is multiplied by a positive function M, this has the effect of replacing P, Q, R, S by MP, MQ, MR, MS, respectively. Both the hypothesis and the conclusion are altered accordingly.

Remark 3. Instead of $R' \leqslant 0$ and $S' \leqslant 0$ we could assume only that R and S are weakly decreasing. For proof let $0 \leqslant a < b$. We want to show $z(b) \leqslant z(a)$. Suppose on the contrary that $z(b) = z(a) + 2\delta$ where $\delta > 0$. Since

$$z(t) \geqslant P(t)\, y'(t)^2 + R(b)\, y(t)^2 + \frac{S(b)}{y(t)^2}, \qquad a \leqslant t \leqslant b,$$

and the right-hand side is continuous, we have $z(t) \geqslant z(a) + \delta$ on some interval $[c, b]$, $a < c < b$. On $[a, b]$ we approximate R and S within ε in the L_1 norm by decreasing functions R_ε and S_ε of class C^1. Let a corresponding solution y_ε of the differential equation have the same initial values $y(a)$, $y'(a)$ as y. On the one hand $y_\varepsilon \to y$ and $y'_\varepsilon \to y'$ uniformly on $[a, b]$ as $\varepsilon \to 0$ and on the other hand the corresponding function $z_\varepsilon(t)$ satisfies $z_\varepsilon(t) \leqslant z_\varepsilon(a)$ on $[a, b]$ by Lemma 5. Thus

$$\int_c^b |z - z_\varepsilon|\, dt = \int_c^b (z - z_\varepsilon)\, dt \geqslant \delta(b - c)$$

which contradicts the fact that the integral on the left approaches 0 with ε.

Suppose r satisfies (a) in its original form, without the assumption $p=1$, and set $r=vy$ where $v>0$ and $v\in C^3(I)$. If $r\in C^2$ the resulting equation

$$pvy''+(2pv'+p'v)\,y'+[(pv')'+qv]\;y=\frac{c^2}{pv^3y^3} \tag{6}$$

has the form of that in Lemma 5. However, for present purposes the equation should be multiplied by pv^3 to give

$$P=p^2v^4, \qquad Q=2p^2v^3v'+pp'v^4, \quad R=pv^3(pv')'+pqv^4, \quad S=c^2.$$

Hence $P'=2Q$, $S'=0$, and the sole remaining hypothesis of Lemma 5 is $R'\leqslant 0$. By Remark 3 this can be replaced by $R\downarrow$. Taking $v=(pq)^{-1/4}$ we obtain $R=(p/q)\{p,q\}+1$ where

$$\{p,q\}=\frac{(pv')'}{pv}=\frac{1}{16}\left(\left(\frac{p'}{p}\right)^2-2\frac{p'}{p}\frac{q'}{q}+5\left(\frac{q'}{q}\right)^2-4\frac{p''}{p}-4\frac{q''}{q}\right). \tag{7}$$

Summarizing, we obtain from Lemma 5:

THEOREM 4. *On an open interval I let $p,q>0$, $p,q\in C^2(I)$, and*

$$(pr')'+qr=\frac{c^2}{pr^3}, \tag{a}$$

where $c>0$. Set $y=(pq)^{1/4}r$ and $R=(p/q)\{p,q\}+1$. If $R\downarrow$ then $(Ry^2+c^2y^{-2})\downarrow$ on the sequence of points t_k where $y'(t_k)=0$, hence at the interior extrema of y.

COROLLARY 3. *In addition to the assumption $R\downarrow$, suppose that $R_0=\lim_{t\to\infty}R(t)>0$. Then $Y_0=\lim_{k\to\infty}y(t_{2k})$ and $Y_1=\lim_{k\to\infty}y(t_{2k+1})$ exist, and $Y_0Y_1=c/\sqrt{R_0}$.*

Proof. Theorem 4 proves the existence of

$$\gamma=\lim_{k\to\infty}\,(R(t_k)\,y(t_k)^2+c^2y(t_k)^{-2}).$$

Therefore $R_0y(t_k)^2\leqslant R(t_k)\,y(t_k)^2\leqslant R(t_k)\,y(t_k)^2+c^2y(t_k)^{-2}\to\gamma$, which shows that $\{y(t_k)\}$ is bounded. Thus, since $R(t_k)\to R_0$,

$$R_0\,y(t_k)^2+c^2y(t_k)^{-2}\to\gamma$$

so that $\{y(t_k)\}$ can have only the accumulation points

$$Z_0 = \left(\frac{\gamma - \sqrt{\gamma^2 - 4c^2 R_0}}{2R_0}\right)^{1/2} \quad \text{and} \quad Z_1 = \left(\frac{\gamma + \sqrt{\gamma^2 - 4c^2 R_0}}{2R_0}\right)^{1/2}.$$

These clearly satisfy $Z_0 Z_1 = c/\sqrt{R_0}$. The assertion is therefore proved if $Z_0 = Z_1$. Otherwise $Z_0 < Z_1$, which implies $Z_0^2 < c/\sqrt{R_0} < Z_1^2$. At the extreme points $y' = 0$ and (6) takes the form

$$p^2 v^4 y y'' = \frac{c^2}{y^2} - R y^2 = \frac{c^2}{y^2} - R_0 y^2 + o(1).$$

This shows that $y''(t_k) > 0$ (for large k) if $y(t_k)$ is close to Z_0, and $y''(t_k) < 0$ if $y(t_k)$ is close to Z_1. Thus, at Z_0 only relative minima $y(t_k)$ can accumulate and at Z_1 only relative maxima. The assertion follows since minimum and maximum points interlace.

6. CONNECTIONS WITH A THEOREM OF WINTNER

Under the hypothesis of Corollary 3 with $p = 1$, the relative maxima and minima of $y = q^{1/4} r$ tend to nonzero constants as $t \to \infty$. Results of this kind are connected with the following theorem of Wintner:

THEOREM A (Wintner, 1947). *Let $y'' + qy = 0$ on $[0, \infty)$ where $q \in C^2$ and $q > 0$. Define*

$$\{q\} = \frac{(q^{-1/4})''}{q^{-1/4}} = \frac{5}{16}\left(\frac{q'}{q}\right)^2 - \frac{1}{4}\left(\frac{q''}{q}\right) = \{1, q\}$$

and suppose

$$\int_0^\infty \frac{|\{q\}(t)|}{\sqrt{q(t)}}\, dt < \infty, \qquad \phi(t) = \int_0^t \sqrt{q(\tau)}\, d\tau, \quad \phi(\infty) = \infty.$$

Then if α and β are any constants, there exists one and only one solution y such that $Y = q^{1/4} y$ satisfies the asymptotic relations

$$Y(t) = (\alpha + o(1)) \cos \phi(t) + (\beta + o(1)) \sin \phi(t),$$
$$Y'(t)\, q(t)^{-1/2} = -(\alpha + o(1)) \sin \phi(t) + (\beta + o(1)) \cos \phi(t).$$

This is the way the theorem is worded in [2]. However, the proof gives the further information that if y is any given solution, there exists a pair of constants α, β such that the above asymptotic formulas hold. Both results are used here. Uniqueness shows that if $\alpha = \beta = 0$ then $y = 0$, so this case can be discarded. Existence of α, β for given y is needed below.

To see the connection with the asymptotic behavior of maxima, let W be the class of real-valued functions Y that are defined for all large t and have one of the following two properties:

(i) The relative maxima are isolated and form an infinite sequence $Y(t_k)$, $k = 1, 2, 3, \ldots$, and $\lim_{k\to\infty} Y(t_k) = c$ where $c = c(Y)$ is constant.

(ii) $\lim_{t\to\infty} Y(t) = c$ where $c = c(Y)$ is constant.

The corresponding definition for minima is expressed by $-Y \in W$. The most important property is (i), but we need (ii) to account for certain exceptional cases.

Clearly $\sin t \in W$, but $(1 + o(1)) \sin t$ need not be in W. For example, the function

$$Y(t) = (1 + t^{-1} \cos t^3) \sin t \tag{8}$$

satisfies $Y'(t) = \cos t - 3t \sin t \sin t^3 + O(1/t)$. Using an abbreviated but self-explanatory notation, consider values $t \to \infty$ on which $|t - m\pi| > 0.001$, say. Then $Y'(t)$ changes sign near each value $t = (n\pi)^{1/3}$ and hence the maxima are not asymptotically constant.

In view of this example, the proof that $Y \in W$ in Wintner's theorem requires a bit of care. Let α and β be constants, not both 0, and let ϕ be real. Then

$$\alpha \cos \phi + \beta \sin \phi = \rho \cos(\phi - \gamma) \tag{9}$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$, $\cos \gamma = \alpha/\rho$, $\sin \gamma = \beta/\rho$. Interpreting γ as the angle formed by a radius to a point on the unit circle, we see that γ has a regular behavior as a function of (α, β). Hence

$$(\alpha + o(1)) \cos \phi + (\beta + o(1)) \sin \phi = A_1 \cos(\phi - B_1),$$

where $A_1 \to \rho$ and $B_1 \to \gamma$ as $t \to \infty$.

As we have already observed, this does not lead to the class W. But if we first differentiate the identity (8) with respect to ϕ and then introduce terms $o(1)$, we get

$$-(\alpha + o(1)) \sin \phi + (\beta + o(1)) \cos \phi = -A_2 \sin(\phi - B_2),$$

where $A_2 \to \rho$ and $B_2 \to \gamma$ as $t \to \infty$. In this notation, the equation $Y' = 0$ in Theorem A entails $\sin(\phi - B_2) = 0$. From

$$(\phi - B_1) - (\phi - B_2) = B_2 - B_1 \to \gamma - \gamma = 0,$$

it follows that $\cos(\phi - B_1)$ is close to ± 1 at the relative extrema of Y. Since A_1 has a positive limit we conclude that Y, $-Y$, and $|Y|$ are all in W.

The functions $U = \sin t$ and $V = \sin \sqrt{2}\, t$ show that U and V can be in W while $\sqrt{U^2 + V^2}$ is not. Nevertheless $q^{1/4} r \in W$ when the hypothesis $q^{1/4} u$, $q^{1/4} v \in W$ is obtained from Theorem A. To see why, let u and v be linearly independent solutions of $y'' + qy = 0$. Wintner's theorem without the $o(1)$ terms would give

$$\xi = \alpha \cos \phi + \beta \sin \phi, \qquad \eta = \gamma \cos \phi + \delta \sin \phi,$$

where $\xi = q^{1/4} u$, $\eta = q^{1/4} v$, and α, β, γ, δ are real constants. If $\alpha\delta \neq \beta\gamma$, the equation

$$A\xi^2 + 2B\xi\eta + C\eta^2 = 1$$

leads to an equation for the constants A, B, C with determinant

$$\begin{vmatrix} \alpha^2 & 2\alpha\gamma & \gamma^2 \\ \alpha\beta & \alpha\delta + \beta\gamma & \gamma\delta \\ \beta^2 & 2\beta\delta & \delta^2 \end{vmatrix} = (\alpha\delta - \beta\gamma)^3.$$

Since the determinant is nonzero, the equations have a solution and the (ξ, η) locus is a conic. Being bounded, it is an ellipse.

If $\alpha\delta = \beta\gamma$ there are constants c, d, not both zero, such that $c\xi + d\eta = 0$. In this case the ellipse degenerates to a segment. In either case $q^{1/4} r$ is the distance from the origin to the point (ξ, η) on the ellipse, hence has the constant-maximum and constant-minimum properties described above. (In the degenerate case, the mimima are 0.) Furthermore $q^{1/4} r$ has two maxima and two minima in an interval containing just a single maximum and minimum of ξ or η. This behavior has been confirmed in numerical examples.

Since the ellipse described above is obtained only in the limit, the true curve might introduce extra oscillations analogous to those in (8). To deal with this problem, we introduce the formulas

$$\xi = A_1 \cos(\phi - B_1), \qquad q^{-1/2} \xi' = -A_2 \sin(\phi - B_2),$$

where $A_1 \to \rho_1$, $A_2 \to \rho_1$, $B_1 \to \gamma_1$, $B_2 \to \gamma_1$ as in the simpler case discussed above. Similarly

$$\eta = A_3 \cos(\phi - B_3), \qquad q^{-1/2} \eta' = -A_4 \sin(\phi - B_4),$$

where $A_3 \to \rho_2$, $A_4 \to \rho_2$, $B_3 \to \gamma_2$, $B_4 \to \gamma_2$. The equation $(q^{1/4}r)' = 0$ leads to $\xi\xi' + \eta\eta' = 0$. We want to show that the value ϕ so obtained is close to a value it has for the corresponding point on the limiting ellipse. Once this is done the conclusion $q^{1/4}r \in W$ follows much as before.

On the limiting ellipse the equation $\xi\xi' + \eta\eta' = 0$ takes the form

$$\rho_1^2 \sin 2(\phi - \gamma_1) + \rho_2^2 \sin 2(\phi - \gamma_2) = 0.$$

We assume that $\rho_1 \rho_2 \neq 0$; in the contrary case the analysis simplifies. By symmetry we can also assume that $\rho = \rho_1/\rho_2 \geqslant 1$. With $x = 2(\phi - \gamma_1)$ the equation determining ϕ for the limiting case is

$$\rho^2 \sin x + \sin(x + \gamma) = 0,$$

where $\gamma = 2(\gamma_1 - \gamma_2)$. Thus $\tan x = -(\sin \gamma)/(\cos \gamma + \rho^2)$ unless $\gamma = \pm\pi$ and $\rho = 1$.

So far we have dealt only with the limiting equation. But as long as $\gamma \neq \pm\pi$ or $\rho \neq 1$ the value of x depends continuously on (ρ, γ); indeed, its continuity properties are better than those of $\tan x$. This means that the true value of x, and hence of ϕ, is close to the value found here. Hence $q^{1/4}r \in W$. In the excluded case $\gamma = \pm\pi$ and $\rho = 1$, the location of the maximum is indeterminate, but the limiting value of $q^{1/4}r$ is a constant, so $q^{1/4}r \in W$ follows again.

By introducing an orthogonal transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix},$$

where θ is constant, we can arrange that the limiting ellipse has its axes parallel to the coordinate axes, so that the $\xi\eta$ term is missing. Indeed, since $\xi^2 + \eta^2$ is invariant under such a transformation, the class W for $q^{1/4}r$ is invariant too. Furthermore the new functions ξ_1, η_1 satisfy the differential equation, hence admit the asymptotic formulas of Theorem A. It is left for the reader to decide whether this technique simplifies the above analysis.

7. CONCLUDING REMARKS

If $p = 1$ and r satisfies (a), Theorem A together with the above remarks show that the relative maxima and minima of $q^{1/4}r$ each tend to nonzero limits as $t \to \infty$. Corollary 3 gives the same conclusion (even for $p \neq 1$) under different hypotheses. These results are now generalized.

THEOREM 5. *Let $p>0$, $q>0$, $p, q \in C^2[0,\infty)$ and suppose u is a nontrivial solution of $(pu')'+qu=0$. Suppose further that*

$$\int^{\infty} \sqrt{\frac{p}{q}}\,|\{p,q\}|\,dt<\infty,$$

where $\{p,q\}$ is given by (7). *Then the relative maxima $|y(t_k)|$ of $|y|=(pq)^{1/4}\,|u|$ tend to a positive constant as $t_k\to\infty$. If r satisfies* (a), *the relative maxima and minima of $(pq)^{1/4}r$ also tend to constants.*

Proof of Theorem 5. With $\tilde{r}=(pq)^{-1/4}$ we define

$$\tilde{q}=\frac{1}{p\tilde{r}^4}-\frac{(p\tilde{r}')'}{\tilde{r}}=q-p\{p,q\}.$$

Then $(p\tilde{r}')'+\tilde{q}\tilde{r}=1/p\tilde{r}^3$ and Theorem 1 shows that the solutions $\tilde{u}$ of $(py')'+\tilde{q}y=0$ have the form $\tilde{r}\cos(s-\theta)$ where θ is constant. Hence the maxima $|\tilde{y}(t_k)|$ of $\tilde{y}=(pq)^{1/4}\,\tilde{u}$ form a constant sequence. To pass from $\tilde{u}$ to the solutions u of $(pu')'+qu=0$, we note that the foregoing result gives $\tilde{w}=O((pq)^{-1/4})$ for all solutions $\tilde{w}$ of the equation with $\tilde{q}$. Hence the integral on the left below is dominated by that on the right:

$$\int^{\infty}|\tilde{w}|^2\,|q-\tilde{q}|\,dt,\qquad \int^{\infty}(pq)^{-1/2}\,p|\{p,q\}|\,dt.$$

The result now follows from [2, p. 372, Exercise 8.4 (a)] and from the representation $r=\sqrt{u^2+v^2}$ given by Theorem 1. The details are similar to those for the case $p=1$ corresponding to Theorem A.

EXAMPLE 1. If $p=a_1e^{at}$ and $q=b_1e^{bt}$ where $a_1>0$, $b_1>0$, a, b are constants, it is easily checked that

$$\{p,q\}=\{e^{at},e^{bt}\}=-\tfrac{1}{16}(a+b)(3a-b).$$

This vanishes for $b=-a$ or $b=3a$. Otherwise we need $a<b$ in the criterion of Theorem 5. When $\{p,q\}=0$ we have $(pr')'=0$ where $r=(pq)^{-1/4}$, so Eq. (a) holds with $c=1$. Hence Theorem 1 shows that the equations on the left below have the solutions on the right when $a\neq 0$:

$$(e^{at}u')'+e^{-at}u=0,\qquad u=\cos\frac{e^{-at}}{a}\quad\text{or}\quad\sin\frac{e^{-at}}{a},$$

$$(e^{at}u')'+e^{3at}u=0,\qquad u=e^{-at}\cos\frac{e^{at}}{a}\quad\text{or}\quad e^{-at}\sin\frac{e^{at}}{a}.$$

EXAMPLE 2. If $p = a_1 t^a$ and $q = b_1 t^b$ with constants as in Example 1, we have

$$\{p, q\} = \{t^a, t^b\} = -\frac{(a+b)(3a-b-4)}{16t^2}.$$

This vanishes when $b = -a$ or $b = 3a-4$. Otherwise we need $a - b < 2$ to get the conclusion of Theorem 5. If $a \neq 1$, the equations on the left below have the solutions on the right:

$$(t^a u')' + t^{-a} u = 0, \qquad u = \cos \frac{t^{1-a}}{1-a} \quad \text{or} \quad \sin \frac{t^{1-a}}{1-a},$$

$$(t^a u')' + t^{3a-4} u = 0, \qquad u = t^{1-a} \cos \frac{t^{a-1}}{a-1} \quad \text{or} \quad t^{1-a} \sin \frac{t^{a-1}}{a-1}.$$

ACKNOWLEDGEMENTS

We are indebted to Professors Roland Lemmert and Johann Walter for valuable orientation regarding the literature. Redheffer's part of this research was carried out at the Mathematisches Institut I, University of Karlsruhe, under auspices of the U.S. Special Program, Alexander von Humboldt-Stiftung.

REFERENCES

1. M. S. P. Eastham, "The Asymptotic Solution of Linear Differential Systems," Clarendon, Oxford, 1989.
2. P. Hartman, "Ordinary Differential Equations," Wiley, New York, 1964.
3. R. Redheffer, Steen's equation and its generalizations, *Aequationes Math.* **57** (1999), 1–13.
4. R. Redheffer, Monotonicity of the maxima, *Amer. Math. Monthly* **105** (1998), 945–948.
5. R. Redheffer, A remarkable differential equation, Proceedings of the Plovdiv Conference on Differential Equations, 1998.
6. I. Redheffer and R. Redheffer, Steen's 1874 paper, historical survey and English translation, to appear.
7. G. Szegő, "Orthogonal Polynomials," Amer. Math. Soc., New York, 1959.
8. J. Walter, Bemerkungen zu dem Grenzpunktfallkriterium von N. Levinson, *Math. Z.* **105** (1968), 345–350.